SUGGESTED SOLUTION TO HOMEWORK 1

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Problem 1. Let X and Y be two normed space. Show that a linear operator is bounded if and only if it maps bounded sets in X into bounded subsets of Y.

Proof. $\Leftarrow$:Let T be a linear operator that maps bounded sets in X into bounded subsets of Y. For arbitrary $0 \neq x \in X$, we denote

$$\bar{x} := \frac{x}{\|x\|_X},$$

then $\|\bar{x}\|_X \leq 1$, therefore $\bar{x} \in B_X(0,1)$. Since $T(B_X(0,1)) \subset Y$ is bounded, without loss of generality, we assume there exists a constant $r > 0$ such that $T(B_X(0,1)) \subset B_Y(0,r)$, therefore

$$\|T(\bar{x})\|_Y \leq r,$$

by the linearity of T,

$$\|T(x)\|_Y \leq r\|x\|_X,$$

which implies T is bounded.

$\Rightarrow$:Let T be a linear bounded operator from X to Y. Then

$$\|T(x)\|_Y \leq \|T\| \cdot \|x\|_X, \quad \forall x \in X.$$

For arbitrary bounded set $U \subset X$, there exists $R > 0$ such that $U \subset B_X(0,R)$, then for arbitrary $x \in U$,

$$\|T(x)\|_Y \leq R\|T\|,$$

which implies $T(U) \subset B_Y(0, R\|T\|)$ is a bounded set. □

Problem 2. Prove that operators of the left and right shift on ℓ_p are bounded and $\|T_l\| = \|T_r\| = 1$.

Proof. For the left shift operator T_l,

$$\|T_l(x)\|_p^p = \sum_{i=1}^{p} |x(i+1)|^p \leq \sum_{j=1}^{p} |x(j)|^p = \|x\|_p^p, \quad \forall x \in \ell_p,$$

which implies $\|T_l\| \leq 1$. Moreover, since for $x = e_2 = (0,1,0,\cdots) \in \ell_p$,

$$T(e_2) = e_1 = (1,0,\cdots),$$

therefore $\|T(e_2)\|_p = 1$, $\|e_2\|_p = 1$, which implies $\|T_l\| = 1$.

Similarly, we can prove $\|T_r\| = 1$. □

Problem 3. Let T be a bounded operator from a normed space X to a normed space Y. Prove that for every $x \in X$ and $r > 0$,

$$\sup_{y \in B(x,r)} \|Ty\| \geq \|T\|r.$$

Proof. Since

$$\|T\| = \sup_{\|y\|=r} \frac{\|Ty\|}{r},$$

then

$$r\|T\| \le \sup_{\|y\|\le r} \|Ty\|.$$

Note that by the triangle inequality,

$$2\|Ty\| \le \|Ty - Tx\| + \|Ty + Tx\|,$$

therefore

$$\|Ty\| \le \max\{\|Ty - Tx\|, \|Ty + Tx\|\},$$

hence

$$r\|T\| \le \sup_{\|y\|\le r} \max\{\|Ty - Tx\|, \|Ty + Tx\|\} \le \sup_{\|y-x\|\le r} \|Ty\|.$$

□

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